Topic: Number System

Sub-topic: Irrational numbers

Class: IX

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Specific Objective: Children construct the knowledge of irrational numbers from the knowledge of rational numbers, learn to identify rational and irrational numbers.

Previous Knowledge: Whole numbers, rational numbers, law of exponents.

The Dangerous Ratio

It's a stormy day on the sea off the coast of Greece. The date is around 520 BC. Fighting for his life, a man is heaved over the side of a boat and dropped into the open water to die. His name is Hippasus of Metapontum.

His crime?

Telling the world a mathematical secret. The secret of the dangerous ratio.

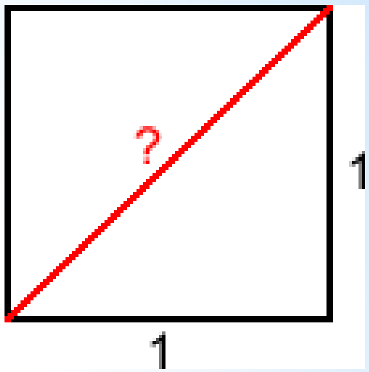
The murder of Hippasus is a matter of legend, but the secret was real, and certainly dangerous enough to the beliefs of those who knew about it.

It was a secret owned by the school of Pythagoras. Early Greek mathematicians (Pythagoras himself was born around 569 BC) were obsessed with the significance of whole numbers and their ratios. The Pythagorean's motto, carved above the entrance of the school, was "All is number"(by which they meant whole numbers).

The Pythgorean school believed that the universe was built around the whole numbers. Each number from one to ten was given a very special significance. Odd numbers were thought to be male and even numbers female. Yet there was one number that the Pythagoreans found terrifying, the number that might have cost Hippasus his life for revealing its existence to the world.

The name Pythagoras these days is best remembered for a geometrical theorem, the one that tells us that in a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides, and it is from this theorem that the dangerous ratio emerges.

Let us consider a square, each side 1 unit in length. How long is the square's diagonal?



This seemingly harmless question was the trigger for the Pythagoreans' disturbing discovery.

By Pythagoras' theorem,

Diagonal = $\sqrt{1^2 + 1^2}$ = $\sqrt{2}$

i.e. the length of the diagonal is the number which when multiplied by itself makes 2.

But what is that number?

The square root of 2 isn't 1 because 1x1 is 1.

And it isn't 2, because 2x2 is 4.

It's something in between.

This wasn't a problem for the Pythagoreans. It was obviously a ratio of two whole numbers. They only had to figure out what that ratio was. At least that was the theory.

But after more and more frantic attempts, a horrible discovery was made. There is NO ratio that will produce $\sqrt{2}$ - it simply can't be done. It's what we now call an irrational number, not because it is illogical, but because it can't be represented as a ratio of whole numbers.

This was what sent the Pythagoreans into such a spin that they may have sacrificed poor Hippasus. If you believe that everything is constructed from whole numbers, it is a terrible a shock to discover that there is an everyday number, a 'real world' number like the diagonal of a square, that doesn't fit your picture of the world. It's a nightmare - and one from which the Pythagoreans would never really recover.

Rational Numbers

Rational Numbers

A rational number is any number that can be expressed as the ratio of two integers.

$$\frac{a}{b}$$

The decimal representation of rational numbers are terminating or non-terminating but repeating.

Examples

$$\frac{4}{5} \qquad 2\frac{2}{3} = \frac{8}{3} \qquad 6 = \frac{6}{1} \qquad -3 = -\frac{3}{1} \qquad 2.7 = \frac{27}{10}$$

$$0.7 = \frac{7}{10} \qquad 0.625 = \frac{5}{8} \qquad 34.56 = \frac{3456}{100}$$

$$0.\dot{3} = \frac{1}{3} \qquad 0.\dot{2}\dot{7} = \frac{3}{11} \qquad 0.\dot{1}4285\dot{7} = \frac{1}{7}$$

So the decimal representation of irrational number is non-terminating and non-repeating. Their decimal expansion form shows no pattern whatsoever.

Examples

0.10110111011110..., 1.12345678910111213…,1.01001000100001…

$\sqrt{2}$ = 1.41421356237309504880168872420 96...
π = 3.14159265358979323846264338327950...

The square roots of numbers that are NOT perfect squares are irrational. e.g. $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, ...

Rational and Irrational Numbers

Combining Rationals and Irrationals

Addition and subtraction of an integer to an irrational number gives another irrational number, as does multiplication and division.

Examples of irrationals

[illegible]

[illegible]

$3+10\sqrt{3}+2$[illegible]

$=28+10\sqrt{3}$

$6+9\sqrt{6}+1$[illegible]

$20+9\sqrt{6}$

Rational and Irrational Numbers

Combining Rationals and Irrationals

Multiplication and division of an irrational number by another irrational can often lead to a rational number.

Examples of Rationals

[illegible]

↓ 21

↓ 26

↓ 8

↓ 1

↓ -13

Rational and Irrational Numbers

Combining Rationals and Irrationals

Determine whether the following are rational or irrational.

(a) 0.73 — rational

(b) $\sqrt{2}$ — irrational

(c) 0.666.... — rational

(d) 3.142 — rational

(e) $\sqrt{1225}$ — irrational

(f) $\sqrt{7}$ — irrational

(g) $4+\sqrt{5}$ — irrational

(h) $(\sqrt[3]{2})^3+1$ — rational

(i) $16^{\frac{1}{2}}$ — rational

(j) $(\sqrt[3]{2})^2$ — irrational

(j) $(\sqrt{3}+1)(\sqrt{3}+1)$ — irrational

(k) $(\sqrt{6}+1)(\sqrt{6}-1)$ — rational

(l) $(1+\sqrt{2})(1-\sqrt{2})$ — rational

Rational and Irrational Numbers

Questions

State whether each of the following are rational or irrational.

a $\sqrt{6} \times \sqrt{7}$ — irrational

b $\sqrt{20} \times \sqrt{5}$ — rational

c $\sqrt{27} \times \sqrt{3}$ — rational

d $\sqrt{4} \times \sqrt{3}$ — irrational

e $\dfrac{\sqrt{32}}{\sqrt{8}}$ — rational

f $\dfrac{\sqrt{44}}{\sqrt{11}}$ — rational

g $\dfrac{\sqrt{18}}{\sqrt{2}}$ — rational

h $\dfrac{\sqrt{25}}{\sqrt{5}}$ — irrational

Thank You